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An asymptotic theory of clad inhomogeneous planar waveguides: II. Solutions of the eigenvalue equation

J M Arnold

Department of Electrical and Electronic Engineering, University of Nottingham, University Park, Nottingham, NG7 2RD, UK

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Abstract. The eigenvalue problem for the differential equation describing scalar waves in a clad inhomogeneous planar waveguide is solved using an asymptotic formulation derived in the preceding paper. The results of this earlier paper are summarised, and a method proposed for the extraction of the eigenvalue from the implicit eigenvalue equation. Although in general these calculations are conveniently carried out by numerical methods, approximate closed-form expressions can be obtained for the eigenvalue and these are given for all the asymptotic regimes of interest.

1. Introduction

This paper is a continuation of the preceding one (Arnold 1980a, hereafter referred to as I) in which the determination to *all asymptotic orders* of the eigenfunctions and eigenvalue equations of scalar waves in certain planar waveguiding environments was carried out. The waveguides in question are transversely inhomogeneous with symmetric inhomogeneity profiles, embedded in symmetric homogeneous cladding media. The problem to be considered here is that of the solution of the eigenvalue equations derived in I, to describe the entire discrete part of the spectrum.

Section 2 contains a summary of all the relevant expressions derived in I, and § 3 is devoted to the techniques for solving the eigenvalue equations, along with approximate expressions derived in various asymptotic regimes. These explicit expressions are obtained by retaining only the leading-order terms of asymptotic series as the large parameter V becomes infinite. It should be emphasised here that this low order of approximation is not an essential restriction; if more terms are retained then the complexity of the resulting expressions increases greatly, without increasing significantly their *physical* content. For numerical accuracy, however, as many asymptotic terms as desired may be retained and all such terms can be generated from the theory described here and in I.

The resulting formulae for the eigenvalues are exhibited as being asymptotic to their zero-order WKB values, which latter neglect the effects of the cladding boundaries; corrections are given both for higher-order WKB effects (Froman 1970) and effects due to the finite boundary. The zero-order WKB values are implicitly defined, and can be rendered explicit by a variety of devices. The method considered here involves regarding the profile function f^2 as a small perturbation on an ideal quadratic function of the transverse coordinate, but others are possible.

2. Summary of asymptotic expressions

In this section the necessary results from I will be summarised.

The differential equation is

$$\mathrm{d}^2\phi/\mathrm{d}x^2-(V^2f^2-U^2)\phi=0 \tag{2.1a}$$

subject to boundary conditions

$$\frac{1}{\phi}\frac{\mathrm{d}\phi}{\mathrm{d}x}=\pm W \qquad x=\pm 1 \tag{2.1b}$$

where f^2 is any suitable analytic function such that $f=0$ at $x=0$ and $f=1$ at $x=1$. In addition, in I f was assumed to be symmetrical about $x=0(f(x)=-f(-x))$ although this restriction is not essential; it simplifies some of the calculations. Acceptable forms for f (or f^2) include:

$$f=x\sum_{j=0}^{M}a_jx^{2j} \qquad \sum_{j=0}^{M}a_j=1, \qquad a_0=1 \tag{2.2a}$$

and

$$f^2=x^2\sum_{j=0}^{M}b_jx^{2j} \qquad \sum_{j=0}^{M}b_j=1, \qquad b_0=1. \tag{2.2b}$$

The parameter W is given by

$$W^2=V^2-U^2. \tag{2.3}$$

It is further assumed that $V^2f^2-U^2$ has only two zeros, x_1 and x_2 (such that $x_2=-x_1$ and $x_2>0$), in $-1<x<1$ for $U^2\leqslant V^2$. For convenience we also write

$$F^2=f^2-f_0^2 \tag{2.4a}$$

$$f_0^2=U^2/V^2. \tag{2.4b}$$

The solution ϕ of (2.1) then has various representations.

2.1. Evanescent WKB

$$\phi=(\mathrm{d}\zeta/\mathrm{d}x)^{-1/2}(A_1\,\mathrm{e}^{-V\zeta}+A_2\,\mathrm{e}^{V\zeta}) \tag{2.5}$$

where

$$\left(\frac{\mathrm{d}\zeta}{\mathrm{d}x}\right)^2=w^2=F^2-\frac{1}{V^2}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}x}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}x}\right)^{-1/2} \tag{2.6}$$

$$\zeta=\tfrac{1}{2}\int_\Gamma w\,\mathrm{d}x' \tag{2.7}$$

and Γ is a contour in the complex x'-plane which starts and finishes at x after passing once around the turning point $x'=x_2$ in a clockwise direction. The constants A_1 and A_2 are

$$A_1=A_1^{\mathrm{e}}=A_1^{\mathrm{e}'}\cos(\nu\pi/2) \tag{2.8a}$$

$$=A_1^{\mathrm{o}}=A_1^{\mathrm{o}'}\sin(\nu\pi/2) \tag{2.8b}$$

$$A_2=A_2^{\mathrm{e}}=-2A_2^{\mathrm{e}'}\sin(\nu\pi/2) \tag{2.8c}$$

$$=A_2^{\mathrm{o}}=2A_2^{\mathrm{o}'}\cos(\nu\pi/2) \tag{2.8d}$$

where the superscripts refer to solutions with even or odd symmetry about $x=0$. In (2.8), ν is given by

$$\nu+\tfrac{1}{2}=-\frac{V}{2\pi \mathrm{i}}\int_{\Gamma_0} w\,\mathrm{d}x \tag{2.9}$$

where Γ_0 encircles the two turning points $x=x_1$ and $x=x_2$ in a positive (counter-clockwise) sense, and w is a solution of (2.6). The values of the constants $A_1^{e'}$, $A_2^{e'}$, $A_1^{o'}$, $A_2^{o'}$ are given in I, and only the ratios

$$A_2^{e'}/A_1^{e'}=1 \tag{2.10a}$$

$$A_2^{o'}/A_1^{o'}=1 \tag{2.10b}$$

are required to determine the eigenvalues.

The above expressions apply for $x>x_2$, and are exact, in the sense that they are valid for *any* asymptotic order of approximation to w from (2.6).

The asymptotic solution of (2.6) is

$$w\sim F-\frac{1}{2V^2F}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{4F^2}\frac{\mathrm{d}F^2}{\mathrm{d}x}\right)+\left(\frac{1}{4F^2}\frac{\mathrm{d}F^2}{\mathrm{d}x}\right)^2\right]+\mathrm{O}(V^{-4}) \tag{2.11}$$

and higher-order terms follow by iteration of (2.6).

Expressions for $x<x_1$ are obtained by straightforward symmetry considerations.

2.2. *Oscillatory WKB*

$$\phi=(\mathrm{d}\eta/\mathrm{d}x)^{-1/2}(B_1\,\mathrm{e}^{-\mathrm{i}V\eta}+B_2\,\mathrm{e}^{\mathrm{i}V\eta}) \tag{2.12}$$

for $x_1<x<x_2$, where

$$\left(\frac{\mathrm{d}\eta}{\mathrm{d}x}\right)^2=u^2=-F^2+\frac{1}{V^2}\left(\frac{\mathrm{d}\eta}{\mathrm{d}x}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\mathrm{d}\eta}{\mathrm{d}x}\right)^{-1/2} \tag{2.13}$$

$$\eta=\tfrac{1}{2}\int_{\Gamma} u\,\mathrm{d}x'. \tag{2.14}$$

The constants are

$$B_1=B_1^{e}=B^{e}\,\mathrm{e}^{\mathrm{i}\nu\pi/2}\,\mathrm{e}^{\mathrm{i}\pi/4} \tag{2.15a}$$

$$=B_1^{o}=B^{o}\,\mathrm{e}^{\mathrm{i}\nu\pi/2}\,\mathrm{e}^{\mathrm{i}\pi/4} \tag{2.15b}$$

$$B_2=B_2^{e}=B^{e}\,\mathrm{e}^{-\mathrm{i}\nu\pi/2}-\mathrm{e}^{-\mathrm{i}\pi/4} \tag{2.15c}$$

$$=B_2^{o}=B^{o}\,\mathrm{e}^{-\mathrm{i}\nu\pi/2}\,\mathrm{e}^{-\mathrm{i}\pi/4} \tag{2.15d}$$

$$B^{e}=A_1^{e'} \tag{2.16a}$$

$$B^{o}=A_1^{o'}. \tag{2.16b}$$

The asymptotic solution of (2.13) is

$$u\sim(-F^2)^{1/2}+\frac{(-F^2)^{-1/2}}{2V^2}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{4F^2}\frac{\mathrm{d}F^2}{\mathrm{d}x}\right)+\left(\frac{1}{4F^2}\frac{\mathrm{d}F^2}{\mathrm{d}x}\right)^2\right]+\mathrm{O}(V^{-4}). \tag{2.17}$$

2.3. Near caustics

$$\phi=(\mathrm{d}\tau/\mathrm{d}x)^{-1/2}[C_1\,\mathrm{Ai}(V^{2/3}\tau)+C_2\,\mathrm{Bi}(V^{2/3}\tau)] \tag{2.18}$$

where

$$\tau\left(\frac{\mathrm{d}\tau}{\mathrm{d}x}\right)^2=F^2-\frac{1}{V^2}\left(\frac{\mathrm{d}\tau}{\mathrm{d}x}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\mathrm{d}\tau}{\mathrm{d}x}\right)^{-1/2} \tag{2.19}$$

$$C_1=C_1^{\mathrm{e}}=C_1^{\mathrm{e}'}\cos(\nu\pi/2) \tag{2.20a}$$

$$=C_1^{\mathrm{o}}=C_1^{\mathrm{o}'}\sin(\nu\pi/2) \tag{2.20b}$$

$$C_2=C_2^{\mathrm{e}}=-C_2^{\mathrm{e}'}\sin(\nu\pi/2) \tag{2.20c}$$

$$=C_2^{\mathrm{o}}=C_2^{\mathrm{o}'}\cos(\nu\pi/2) \tag{2.20d}$$

and

$$C_1^{\mathrm{e}'}=2\pi^{1/2}V^{1/6}A_1^{\mathrm{e}'} \tag{2.21a}$$

$$C_2^{\mathrm{e}'}=2\pi^{1/2}V^{1/6}A_2^{\mathrm{e}'} \tag{2.21b}$$

$$C_1^{\mathrm{o}'}=2\pi^{1/2}V^{1/6}A_1^{\mathrm{o}'} \tag{2.21c}$$

$$C_2^{\mathrm{o}'}=2\pi^{1/2}V^{1/6}A_2^{\mathrm{o}'}. \tag{2.21d}$$

The asymptotic solution of (2.19) is

$$\tau=\tau'+\tau_0'/V^2 \tag{2.22}$$

$$\tfrac{2}{3}\tau'^{3/2}\sim\int_{x_2}^{x}F\,\mathrm{d}x'+\frac{1}{2V^2}\int_{x_2}^{x}\frac{1}{F}\left[\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}x'}\left(\frac{1}{F^2}\frac{\mathrm{d}F^2}{\mathrm{d}x'}-\frac{2}{3}\frac{F}{\int_{x_2}^{x'}F\,\mathrm{d}x''}\right)\right.$$
$$\left.-\frac{1}{16}\left(\frac{1}{F^2}\frac{\mathrm{d}F^2}{\mathrm{d}x}-\frac{2}{3}\frac{F}{\int_{x_2}^{x'}F\,\mathrm{d}x''}\right)^2-\tau_0'^{(0)}F^2\left(\frac{3}{2}\int_{x_2}^{x'}F\,\mathrm{d}x''\right)^{-2/3}\right]\mathrm{d}x'+\mathrm{O}(V^{-4}) \tag{2.23}$$

for $x\sim x_2$, where

$$\tau_0'\sim\tau_0'^{(0)}+\mathrm{O}(V^{-2}) \tag{2.24}$$

$$\tau_0'^{(0)}=-\tfrac{3}{10}F_1^{-1/3}F_2 \tag{2.25}$$

$$F_1=\mathrm{d}F^2/\mathrm{d}x|_{x=x_2} \tag{2.26a}$$

$$F_2=\mathrm{d}^2F^2/\mathrm{d}x^2|_{x=x_2}. \tag{2.26b}$$

2.4. Uniform ($-1\leqslant x\leqslant 1$)

$$\phi=(\mathrm{d}\xi/\mathrm{d}x)^{-1/2}[\Phi_1\cos(\nu\pi/2)-\Phi_2\sin(\nu\pi/2)] \tag{2.27}$$

where

$$(\xi^2-\xi_0^2)\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^2=F^2-\frac{1}{V^2}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^{-1/2} \tag{2.28}$$

$$V\xi_0^2=2(\nu+\tfrac{1}{2}) \tag{2.29}$$

and Φ_1 and Φ_2 are linearly independent solutions of

$$d^2\Phi/d\xi^2 - V^2(\xi^2-\xi_0^2)\Phi = 0. \tag{2.30}$$

Integral representations for Φ_1 and Φ_2 are given in I, § 3, equations (3.35)–(3.38).

2.5. *Degenerate caustics*

In general, (2.28) cannot be solved explicitly. However, when $U^2 \sim O(V)$, f_0^2 is small and an asymptotic solution can be constructed explicitly. It is

$$\frac{\xi^2}{2} \sim \int_0^x f\,dx' - \frac{1}{2V}\int_0^x \left(\frac{\chi}{f} - \frac{\chi'}{2}\frac{f}{\int_0^{x'} f\,dx''}\right) dx' + O(V^{-2}) \tag{2.31}$$

where

$$f_0^2 = \chi/V \tag{2.32a}$$

$$\xi_0^2 = \chi'/V \tag{2.32b}$$

and $\chi, \chi' \sim O(1)$ as $V \to \infty$, being chosen so that the second integral in (2.31) converges at $x'=0$.

Depending on the locations of the two caustics, there is a variety of eigenvalue equations.

(i) *Generic form*

The eigenvalue equation always takes the form

$$-\frac{V}{2\pi i}\oint_{\Gamma_0} w\,dx = q + \frac{1}{2} + \frac{2\theta}{\pi} \tag{2.33}$$

where q is a non-negative integer; the expression for θ is dependent on the locations of the caustics. Γ_0 encircles the two turning points counter-clockwise.

(ii) *Caustics well separated from each other and the boundaries*

$$\theta = \tan^{-1}\left[\frac{1}{2}\frac{(w^{1/2}\,dw^{-1/2}/dx - Vw + W)}{(w^{1/2}\,dw^{-1/2}/dx + Vw + W)}\exp\left(-V\int_\Gamma w\,dx'\right)\right] \qquad x=1 \tag{2.34}$$

with w defined by (2.6), at $x=1$.

(iii) *Caustic near a boundary*

$$\theta = \tan^{-1}\left(\frac{\mathrm{Ai}'(V^{2/3}\tau) + V^{-2/3}W''\,\mathrm{Ai}(V^{2/3}\tau)}{\mathrm{Bi}'(V^{2/3}\tau) + V^{2/3}W''\,\mathrm{Bi}(V^{2/3}\tau)}\right) \qquad x=1 \tag{2.35}$$

where

$$W'' = \left(\frac{d\tau}{dx}\right)^{-1}\left[W + \left(\frac{d\tau}{dx}\right)^{1/2}\frac{d}{dx}\left(\frac{d\tau}{dx}\right)^{-1/2}\right] \tag{2.36}$$

and τ is defined by (2.19), at $x=1$.

(iv) *Degenerate caustics*

$$\theta = \tan^{-1}\left(\frac{(\Phi_1^{e'} + W'''\Phi_1^{e})}{(\Phi_2^{e'} + W'''\Phi_2^{e})}\right) \qquad x=1 \qquad (q \text{ even}) \tag{2.37a}$$

$$\theta = \tan^{-1}\left(\frac{(\Phi_1^{o'} + W'''\Phi_1^{o})}{(\Phi_2^{o'} + W'''\Phi_2^{o})}\right) \qquad x=1 \qquad (q \text{ odd}) \tag{2.37b}$$

where the prime on the Φ functions signifies differentiation with respect to ξ,

$$W''' = \left(\frac{d\xi}{dx}\right)^{-1}\left[W + \left(\frac{d\xi}{dx}\right)^{1/2}\frac{d}{dx}\left(\frac{d\xi}{dx}\right)^{-1/2}\right] \tag{2.38}$$

evaluated at $x = 1$, and ξ is a solution of (2.28), given asymptotically by (2.31). The various Φ functions are defined in I, § 3.

3. Solution of the eigenvalue equation

The generic equation (2.33) is the eigenvalue equation for all configurations of caustics and is accurate to all asymptotic orders. Intepretations of this equation in terms of better known but less accurate expressions are also possible. If only the leading-order term in w is retained from (2.11), and θ is neglected, (2.33) reduces to

$$\frac{-V}{2\pi i}\int_{\Gamma_0}(f^2 - f_0^2)^{1/2}\,dx \sim q + \tfrac{1}{2} \tag{3.1}$$

which is the leading-order WKB quantisation condition (the Bohr–Sommerfeld condition) in contour integral form. Still neglecting θ, but retaining all terms in w, (2.33) becomes

$$\frac{-V}{2\pi i}\oint_{\Gamma_0} w\,dx \sim q + \tfrac{1}{2} \tag{3.2}$$

which is Dunham's condition (Dunham 1932), expressed slightly differently. Thus, θ represents the effect of finite boundaries, and the higher-order terms in w correct the WKB formulae.

The generic equation (2.33) can be solved formally as follows. Let

$$\nu = q + 2\theta/\pi \tag{3.3}$$

and consider

$$\frac{-V}{2\pi i}\oint_{\Gamma_0} w\,dx = \nu + \tfrac{1}{2} \tag{3.4}$$

which is identical to (2.9). The left-hand side of (3.4) depends on the eigenvalue implicitly through f_0^2 (cf (2.4) and (2.11)). Hence equation (3.4) implicitly defines f_0^2 as a function of ν, which we suppose to be obtainable by inversion of (3.4). This then permits f_0^2, which appears explicitly in θ, to be expressed in terms of ν. Hence, (3.3) acquires the form

$$\nu = q + (2/\pi)\theta(\nu). \tag{3.5}$$

Because of the transcendental functions appearing in the definitions of θ, (2.34), (2.35) and (2.37), it is not possible to obtain simple asymptotic expansions for ν. In general (3.5) is solved by a convenient numerical scheme such as Newton's method. The first iteration of this procedure still yields an analytic expression for ν, and hence f_0^2, which can be used to study the various corrections to WKB which transpire.

Before giving these expressions, it is necessary to consider the inversion of (3.4), which can be done asymptotically, as follows. Equation (3.4) can be rewritten as

$$\frac{1}{2\pi i}\oint_{\Gamma_0} w\, dx = \frac{-p}{2} \tag{3.6}$$

where

$$p = 2(\nu + \tfrac{1}{2})/V \tag{3.7}$$

and is to be regarded as an *arbitrary* parameter. Let f_0^2 be expanded asymptotically as

$$f_0^2 \sim \sum_{j=0}^{\infty} \lambda_j V^{-2j} \tag{3.8}$$

where the $\{\lambda_j\}$ are functions of p to be determined. The integrand in (3.6) is a solution of (2.6), i.e.

$$w^2 \sim f^2 - \lambda_0 - \frac{1}{V^2}\left(w^{1/2}\frac{d^2 w^{-1/2}}{dx^2} + \lambda_1\right) - \sum_{j=2}^{\infty} \lambda_j V^{-2j}. \tag{3.9}$$

Here the higher-order terms of f_0^2 has been used as counterterms (cf I). An asymptotic expansion for w results from iterating (3.9):

$$w^2 \sim F_0^2 - \frac{1}{V^2}\left(F_0^{1/2}\frac{d^2 F_0^{-1/2}}{dx^2} + \lambda_1\right) + O(V^{-4}) \tag{3.10}$$

and higher-order terms follow after further iteration, with

$$F_0^2 = f^2 - \lambda_0. \tag{3.11}$$

Taking the square root of (3.10) yields

$$w \sim F_0 - \frac{1}{2V^2 F_0}\left(F_0^{1/2}\frac{d^2 F_0^{-1/2}}{dx^2} + \lambda_1\right) + O(V^{-4}). \tag{3.12}$$

Equation (3.12) is now integrated over a contour surrounding the two turning points, and powers of V^{-2} matched on either side of (3.6) to give

$$\frac{1}{2\pi i}\oint_{\Gamma_0} (f^2 - \lambda_0)^{1/2}\, dx = -\frac{p}{2} \tag{3.13a}$$

$$\lambda_1 \oint_{\Gamma_0} F_0^{-1}\, dx = -\oint_{\Gamma_0} F_0^{-1/2}\frac{d^2 F_0^{-1/2}}{dx^2}\, dx \tag{3.13b}$$

and similar expressions for higher-order coefficients. (3.13*a*) defines λ_0, (3.13*b*) defines λ_1 in terms of λ_0, and so on.

(3.13*a*) is still implicit for λ_0; one method of rendering it explicit is to take f^2 to be *quasi-quadratic*:

$$f^2 = x^2 + \epsilon g \tag{3.14}$$

where g is an analytic function of x^2 such that $g = 0$ at $x = 0$ and $x = 1$ (for example

$$g = x^2(1 - x^2) \tag{3.15}$$

is suitable), and ϵ is a small number. It is then possible, if ϵ is small enough, to deform the contour Γ_0 slightly so that the expansion of the integrand of (3.13*a*) in powers of ϵ is absolutely and uniformly convergent on the new contour (Γ_0' say). Hence the left-hand side of (3.13a) is an analytic function of ϵ in a neighbourhood of $\epsilon = 0$, and the Lagrange implicit function theorem (Whitaker and Watson 1965) may be used to obtain λ_0 as a convergent series in powers of ϵ. The result is

$$\lambda_0 = p + \frac{\epsilon}{2\pi i} \oint_{\Gamma_0'} (x^2 - p)^{-1/2} g \, dx + O(\epsilon^2) \tag{3.16}$$

for the leading-order terms. The radius of convergence is set by the requirement that $|\epsilon g| < |x^2 - p|$ everywhere within and on Γ_0'. This completes the inversion of (3.3), since f_0^2 is given explicitly by (3.8), (3.13), (3.16) and (3.7). The integral in (3.16) and those appearing in higher-order terms are very easy to calculate. For example, with g given by (3.15), we have

$$\begin{aligned} \frac{1}{2\pi i} \oint_{\Gamma_0'} (x^2 - p)^{-1/2} g \, dx &= \frac{1}{2\pi i} \oint_{\Gamma_0'} (x^2 - p)^{-1/2} x^2 (1 - x^2) \, dx \\ &= \frac{1}{2\pi i} \oint_{\Gamma_0'} x(1 - x^2) \sum_{j=0}^{\infty} c_j \left(-\frac{p}{x^2} \right)^j dx \\ &= -c_1 p - c_2 p^2 \\ &= \tfrac{1}{2} p - \tfrac{3}{8} p^2 \end{aligned} \tag{3.17}$$

where the $\{c_j\}$ are the coefficients in the binomial expansion of $(1 + t)^{-1/2}$ in powers of t. We remark here that for practical purposes Γ_0' may generally be taken to be a circle centred at $x = 0$, with radius slightly, but sufficiently, greater than unity.

The above analysis leads to the conjecture that, if g is a polynomial in x^2, then the coefficients of powers of ϵ in the expansion for λ_0 are polynomials in p, a conjecture which it is straightforward to verify.

It is now possible to proceed with the approximate solution of the eigenvalue equation, considering each caustic configuration in turn.

3.1. Caustics well separated from each other and from the boundaries

Here we require

$$\lim_{V \to \infty} f_0^2 = \text{constant} \neq 0 \text{ or } 1. \tag{3.18}$$

The eigenvalue equation is (2.33) with θ given by (2.34). Under the assumption implied by (3.18), θ is exponentially small. Therefore, a first approximation for ν would be, from (3.5),

$$\nu \sim q \tag{3.19}$$

and hence, from (3.7).

$$p \sim 2(q + \tfrac{1}{2})/V = p_0. \tag{3.20}$$

Using (3.16), with (3.20), gives

$$\lambda_0 \sim p_0 + \frac{\epsilon}{2\pi i} \oint_{\Gamma_0} (x^2 - p_0)^{1/2} g \, dx \tag{3.21}$$

and so, by (3.8),

$$f_0^2 \sim \lambda_0. \tag{3.22}$$

Equation (3.22) is no more than the WKB approximation (to lowest order as $V\to\infty$). To improve it, we use this value of f_0^2 to calculate an approximation for θ using (2.34). If only the *leading-order* term in w is retained, the result is, for large V,

$$\theta \sim \frac{1}{2}\left(1-\frac{8V}{F_1}(1-\lambda_0)^{3/2}\right)^{-1} \exp\left(-2V\int_{x_0}^{1}(f^2-\lambda_0)^{1/2}\,\mathrm{d}x'\right) \tag{3.23}$$

where

$$F_1 = \mathrm{d}F^2/\mathrm{d}x|_{x=1} = \mathrm{d}f^2/\mathrm{d}x|_{x=1} \tag{3.24}$$

and λ_0 is given approximately by (3.21). The parameter x_0 is the zero of the integrand. Using (3.14), a perturbation series in powers of ϵ for the argument of the exponential in (3.21) can be constructed, having the form

$$\int_{x_0}^{1}(f^2-\lambda_0)^{1/2}\,\mathrm{d}x \sim \int_{p_0^{1/2}}^{1}(x^2-p_0)\,\mathrm{d}x+\frac{\epsilon}{2}\int_{p_0^{1/2}}^{1}x^2(g-p_1)(x^2-p_0)^{-1/2}\,\mathrm{d}x+\mathrm{O}(\epsilon^2) \tag{3.25}$$

where the fact that, on the first iteration,

$$\lambda_0 \sim p_0+\epsilon p_1+\mathrm{O}(\epsilon^2)$$

has been used; p_1 is the coefficient of ϵ in (3.21).

An examination of (3.23) reveals that the regimes $\lambda_0 \sim 1$ and $\lambda_0 > 1$ must be avoided to keep θ small and real. This mathematical condition is equivalent to the physical requirement that $V > V_q$ (where V_q is the normalised cut-off frequency for the qth mode), to ensure that $U^2 < V^2$. Avoidance of this regime is secured by the governing condition (3.18).

The approximation (3.23) may now be regarded as an initial estimate for θ *in analytical form*. It may be used in (3.5) to obtain a new estimate for ν, which is then used in (3.4). Subsequent inversion of (3.4) leads to a new value of λ_0, and hence f_0^2, and so on.

It will be noted that no higher-order asymptotic terms of w were used to obtain (3.23), even though these are available from I. This is not a restriction, however, since (3.23) is to be regarded as an initial estimate only, for subsequent numerical iteration. The eigenvalue $U^2(=V^2f_0^2)$ does not have a convenient representation in analytical form when higher-order terms are included, and is regarded merely as a numerical parameter in I.

Having thus obtained a first approximation for θ, (3.23), this can be used in (3.3) and the resulting expression (3.4) solved for f_0^2 to give

$$f_0^2=\frac{U^2}{V^2}\sim\lambda_0+\frac{2}{\pi V\sigma_1}\left(1-\frac{8V}{F_1}(1-\lambda_0)^{3/2}\right)^{-1}\exp\left(-2V\int_{x_0}^{1}(f^2-\lambda_0)^{1/2}\,\mathrm{d}x\right) \tag{3.26}$$

where

$$\sigma_1=\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_0'}(f^2-\lambda_0)^{-1/2}\,\mathrm{d}x \tag{3.27}$$

and λ_0 is the leading-order WKB approximation obtained as a solution of (3.1):

$$\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_0'}(f^2-\lambda_0)^{1/2}\,\mathrm{d}x=-\frac{q+\frac{1}{2}}{V}. \tag{3.28}$$

The parameter x_0 is the positive zero of the integrand in (3.26), and Γ_0' is a deformation of Γ_0 which passes around the two points $f^2 = \lambda_0$ in a positive direction. Equation (3.26) corrects the leading-order WKB approximation to f_0^2 for the presence of finite boundaries; further corrections are obtained by computing the higher-order coefficients $\{\lambda_j : j \geqslant 1\}$ from (3.13) and using them in the asymptotic expansion (3.8).

3.2. *Caustics near boundaries*

In the limit $U \to V$, we have $W = 0$, $f_0^2 = 1$, and the assumptions of the previous section do not apply. This condition corresponds to the approach to cut-off of the waveguide mode under consideration, and this regime requires different approximations for the eigenvalue, which are provided by the representation (2.18) of the eigenfunction ϕ in terms of Airy functions. The associated representation for θ is (2.35).

The same general procedure as in § 3.1 above is used. However, in addition, we introduce a hypothesis to measure the proximity of U to its limiting value V. By estimating the variable τ, assuming that W^2/V^2 is small and that f^2 is locally a quadratic function of x near $x = 1$, it transpires that, at $x = 1$,

$$\tau \sim F_1^{-2/3}(W^2/V^2) + \mathrm{O}[(W^2/V^2)^2] \tag{3.29}$$

where F_1 is given by (3.22). Now when the argument of the Airy functions is sufficiently large, these functions may be replaced by their asymptotic expansions, which contain exponentials, and the eigenvalue equation reduces to the form already considered in § 3.1 above. On the other hand, for W^2 sufficiently small, the argument of these functions is not large enough to justify asymptotic expansion and they must be retained to ensure numerical accuracy. This situation arises if the modulus of the argument $V^{2/3}\tau$ is less than, or nearly equal to, unity; otherwise the large-argument expansions may be used. From (3.29) this happens when

$$W^2/V^2 \leqslant V^{-2/3}F_1^{2/3} \tag{3.30}$$

and this inequality may be regarded as specifying the required smallness of W to justify special treatment; it may be observed that (3.30) implies that W^2/V^2 is $\mathrm{O}(V^{-2/3})$ as $V \to \infty$, a condition which was used in our earlier work (Arnold 1980c).

Subsequent analysis similar to § 3.1 above leads to

$$U^2/V^2 \sim \lambda_0 + (4/\pi V\sigma_1)\tan^{-1}(T) \tag{3.31}$$

where

$$T = \frac{\mathrm{Ai}'(V^{2/3}\tau) + V^{1/3}(1-\lambda_0)^{1/2}\,\mathrm{Ai}(V^{2/3}\tau)}{\mathrm{Bi}'(V^{2/3}\tau) + V^{1/3}(1-\lambda_0)^{1/2}\,\mathrm{Bi}(V^{2/3}\tau)} \tag{3.32}$$

$$\sigma_1 = \frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_0'} (f^2 - \lambda_0)^{-1/2}\,\mathrm{d}x \tag{3.33}$$

$$\tau \sim \left(\frac{3}{2}\int_{x_0}^{x} (f^2 - \lambda_0)^{1/2}\,\mathrm{d}x\right)^{2/3} \tag{3.34}$$

and λ_0 is the solution of

$$\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_0'} (f^2 - \lambda_0)^{1/2}\,\mathrm{d}x = -\frac{(q+\frac{1}{2})}{V} \tag{3.35}$$

which is again the leading-order WKB approximation. The parameter x_0 in (3.34) is the zero of the integrand. An error term $O(V^{-5/3})$ may be added to the right of (3.31) to complete it formally, as this is the estimated magnitude of the neglected terms subject to (3.30).

An explicit solution of (3.35) can be effected in this case *without* further assumptions about f, using (3.30). Since in this regime $\lambda_0 \to 1$ as $V \to \infty$, the integrand on the left-hand side of (3.35) is expanded in powers of $1-\lambda_0(=\lambda_0'$, say) and the resulting series inverted. The result of this operation is

$$\lambda_0 \sim 1 - \Delta/\alpha_1 V \tag{3.36}$$

where

$$\Delta = V - 2(q+\tfrac{1}{2})\alpha_0^{-1} \tag{3.37a}$$

$$-\frac{\alpha_0}{2} = \frac{1}{2\pi i}\oint_{\Gamma_0'} (f^2-1)^{1/2}\,dx \tag{3.37b}$$

$$\alpha_1 = \frac{1}{2\pi i}\oint_{\Gamma_0'} (f^2-1)^{-1/2}\,dx. \tag{3.37c}$$

The particular form of (3.37) is chosen because for quadratic $f^2(f^2=x^2)$, $\alpha_0=\alpha_1=1$; Γ_0' is a contour surrounding $x=\pm 1$, a deformation of the original Γ_0.

Using (3.36), (3.8) and (2.3) in (3.30) it transpires that

$$\Delta \leqslant \alpha_1 F_1^{2/3} V^{-2/3} \tag{3.38a}$$

i.e. using (3.37*a*)

$$V \leqslant 2(q+\tfrac{1}{2})\alpha_0^{-1} + \alpha_1(\alpha_0/2)^{-1/3}F_1^{2/3}(q+\tfrac{1}{2})^{1/3} + O(q^{-1/3}) \tag{3.38b}$$

which sets an upper limit on the value of V for which this type of representation is necessary.

The difference between the expressions given here and those of previous work (Arnold 1980b) (apart from some trivial notation changes such as different meanings for λ_0 and λ_1) is that here the attempt to obtain a formal expansion for the eigenvalue U^2 has been abandoned. Instead, we have chosen to exhibit the parameter U^2/V^2 as a perturbation on its WKB value λ_0, a more convenient and instructive representation. In particular, comparison of the results of this part with those of § 3.1, which is possible directly, indicates that the correction to WKB is relatively *large* in the present case, being $O(V^{-1})$ here in comparison with an *exponentially small* amount in § 3.1. This is a direct consequence of the proximity of the caustic to the boundary.

A further quantity of interest in waveguide theory is the (normalised) cut-off frequency, V_q, of the mode indexed by the integer q. The cut-off condition defines the minimum value of V for which unattenuated propagation of the mode can occur; it corresponds to $W^2=0$, since guided propagation occurs only if W is real and positive (W is the normalised transverse wavenumber in the homogeneous cladding for a transverse dependence of the form e^{-Wx}). Thus, to obtain V_q, W is set to zero and the eigenvalue equation solved for the appropriate value of V.

In this case considerable simplification of the calculation results from setting $W=0$. After some straightforward algebra the result is

$$V_q \sim \Lambda_q - \frac{3^{1/6}}{5\pi}\frac{\Gamma(1/3)}{\Gamma(2/3)}\frac{(F_1^{-4/3}F_2)}{\alpha_0}\Lambda_q^{-2/3} + O(\Lambda_q^{-1}) \tag{3.39}$$

with

$$\Lambda_q = [2(q+\tfrac{1}{2}) - \tfrac{2}{3}]\alpha_0^{-1} \tag{3.40}$$

$$-\frac{\alpha_0}{2} = \frac{1}{2\pi i}\oint (f^2-1)^{1/2}\,dx \tag{3.41}$$

$$F_1 = df^2/dx|_{x=1} \tag{3.42a}$$

$$F_2 = d^2f^2/dx^2|_{x=1}. \tag{3.42b}$$

The term $O(\Lambda_q^{-1})$ in (3.39) can also be quite easily calculated. The corresponding result from the leading-order WKB eigenvalue equation (3.35) would be

$$V_q \sim 2(q+\tfrac{1}{2})\alpha_0^{-1} \tag{3.43}$$

and again it is possible to observe significant corrections to the WKB result by comparing (3.43) with (3.39)–(3.42).

3.3. *Caustics close to each other and to the origin*

This case can be solved directly, using the large-argument asymptotic expansions for the confluent hypergeometric functions in (2.37), along with the explicit solution for ξ as a function of x, given in § 3 of I. However, for all practical purposes the contribution made by θ to (2.33) is extremely small due to the exponential nature of Φ_1^e, Φ_1^o, Φ_2^e and Φ_2^o near the boundary, and for large V it may initially be neglected completely. The eigenvalue problem then reduces to a consideration of the WKB equation to infinite order:

$$-\frac{V}{2\pi i}\oint_{\Gamma_0} w\,dx \sim q+\tfrac{1}{2}. \tag{3.44}$$

On introducing the *ansatz* employed in § 3 of I,

$$f_0^2 \sim O(V^{-1}) \tag{3.45}$$

as $V\to\infty$, solution of (3.44) is equivalent to that of the evanescent wave theory quantisation problem (cf Choudhary and Felsen 1978, Arnold and Felsen 1980). The first few asymptotic terms of the solution to (3.44) are

$$f_0^2 = \frac{U^2}{V^2} \sim \sum_{j=0}^{\infty} \chi_j V^{-j-1} \tag{3.46}$$

$$\chi_0 = 2(q+\tfrac{1}{2}) \tag{3.47a}$$

$$\chi_1 = \frac{-1}{2\pi i}\left(\frac{\chi_0^2}{4}\oint_{\Gamma_0} f^{-3}\,dx + \oint_{\Gamma_0} f^{-1/2}\frac{d^2}{dx^2}f^{-1/2}\,dx\right). \tag{3.47b}$$

With f_0^2 calculated according to these expressions, an estimate for θ can be obtained from (2.37), q replaced by ν according to (3.3), f_0^2 recalculated, and so on.

The fact that the leading-order term

$$U^2/V^2 \sim 2(q+\tfrac{1}{2})/V \tag{3.48}$$

is independent of the precise details of the function f has a physical interpretation;

because both caustics are close to the origin, the propagation behaviour is determined mainly by the local variation of f^2 near $x = 0$, which is (locally) quadratic by hypothesis.

This completes the analysis of the eigenvalue equation.

4. Conclusions

Methods of solving the eigenvalue equations arising in I have been considered for all the asymptotic ($V \to \infty$) regimes of interest for guided waves in symmetric inhomogeneous media immersed in homogeneous claddings.

Because of the transcendental functions which appear in these equations, which must be retained to preserve uniformity, simple expressions in closed form for the eigenvalue cannot be obtained to arbitrary precision. Nevertheless, leading-order approximations in closed form are readily available, and subsequent higher orders of approximation follow by iteration. It should be emphasised that this restriction on closed-form expressions is not a limitation in principle, but is simply a recognition of the complexity of the expressions involved. Since these expressions would have to be evaluated numerically anyway (an operation which would be extremely tedious *because* of the complexity of the expressions), it seems desirable to solve that part of the eigenvalue problem involving transcendental functions (equation (3.5)) directly by numerical means. The separation of the eigenvalue equation into 'asymptotic' and 'transcendental' parts, equations (3.4) and (3.3) respectively, greatly facilitates this procedure. The eigenvalue equation can be said to be solved to given order in V if all the terms up to that order are retained in any function defined by an asymptotic series, and the numerical solution of the resulting approximate eigenvalue equation is carried out to arbitrary numerical precision. We have chosen not to give the higher-order expressions because of their algebraic complexity; they are calculated numerically with no difficulty.

The restriction to symmetric profile functions f and symmetric boundary conditions is inessential; it has been considered desirable in order not to submerge the principles of this method in a welter of calculations. The method is well suited to the treatment of asymmetrical waveguides once the basic principles are understood, and the extension to asymmetrical profiles is to be considered subsequently.

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